SEM-4 (MATHEMATICS GENERAL)
ANUP RAY
SUBGROUP



Theorem 3: The necessary and sufficient conditions that a non-empty sub-set $S$ of a group $G$ froms a subgroup under the binary operation $*$ in $G$ are
(i) $a \in S, b \in S \Rightarrow a * b \in S$
(ii) $a \in S \Rightarrow a^{-1} \in S$, where $a^{-1}$ is the inverse of $a$ in $G$.

Proof Left to Readers. (Home work)

Theorem 4: The necessary and sufficient condition for a non-empty subset of a group $(G, *)$ to be a subgroup is $a \in S, b \in S \Rightarrow a * b^{-1} \in S$, where $b^{-1}$ is the inverse of $b$ in $G$.

Proof: Let $S$ be a subgroup and $a \in S, b \in S$. Since $S$ is a subgroup and $b \in S$, $b^{-1}$ must exist and will belongs to $S$.
now $a \in S, b^{-1} \in S \Rightarrow a * b^{-1} \in S$, by closure property.
Thus the condition in necessary.

To prove that this condition is also sufficient, we assume that
$a \in S, b \in S \Rightarrow a * b^{-1} \in S$
we are to show that $S$ is a subgroup of $G$.
By the given condition, we have
$a \in S, a \in S \Rightarrow a * a^{-1} \in S$ (putting $b = a$)
$\Rightarrow e \in S$
Where $e$ being the identity element.

Again we have $e \in S, a \in S \Rightarrow e a^{-1} \in S$
$\Rightarrow a^{-1} \in S$ where $a^{-1}$ is the inverse of $a$.
now $b \in S$ then $b^{-1} \in S$

Also $a \in S, b^{-1} \in S \Rightarrow a * (b^{-1})^{-1} \in S$

$\Rightarrow a * b \in S$

Now $S \subset G$ and the associative law holds good for $G$, as $G$ is a group. Hence it is true for all the elements of $S$. Thus all Postulates for a group are satisfied for $S$. Hence $S$ is a Subgroup of $G$.

NOTE: In additive composition, the above condition becomes $a \in S, b \in S \Rightarrow a-b \in S$

Home work

1) In each cases determine whether $H$ is a subgroup of the group $G$

a) $H=\{-1, 1\}$, $G=Z$, the additive group of all integers.

b) $H=\{2n / n \in Z\}$, $G=Z$, the additive group of all integers

2) Let $H$ be a subgroup of a group $G$. Show that for any $g \in G$, $K = gHg^{-1}$ $=\{ghg^{-1} / h \in H\}$ is a subgroup of $G$.